

On the Scalar Manifold of Exceptional Supergravity

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Abstract

We construct two parametrizations of the non compact exceptional Lie group $G = E_{7(-25)}$, based on a fibration which has the maximal compact subgroup $K = \frac{E_6 \times U(1)}{\mathbb{Z}_3}$ as a fiber. It is well known that G plays an important role in the $\mathcal{N} = 2$ $d = 4$ magic exceptional supergravity, where it describes the U-duality of the theory and where the symmetric space $\mathcal{M} = \frac{G}{K}$ gives the vector multiplets' scalar manifold.

First, by making use of the exponential map, we compute a realization of $\frac{G}{K}$, that is based on the E_6 invariant d -tensor, and hence exhibits the maximal possible manifest $[(E_6 \times U(1))/\mathbb{Z}_3]$ -covariance. This provides a basis for the corresponding supergravity theory, which is the analogue of the Calabi-Vesentini coordinates.

Then we study the Iwasawa decomposition. Its main feature is that it is $SO(8)$ -covariant and therefore it highlights the role of triality. Along the way we analyze the relevant chain of maximal embeddings which leads to $SO(8)$.

It is worth noticing that being based on the properties of a “mixed” Freudenthal-Tits magic square, the whole procedure can be generalized to a broader class of groups of type E_7 .

1 The “mixed” magic square and the 56 of the Lie algebra $\mathfrak{e}_{7(-25)}$

Exceptional Lie groups act as symmetries in many physical systems. In particular, non compact forms of the group E_7 enter as U-duality of $d = 3$ and $d = 4$ supergravity theories. Here we focus on the $\mathcal{N} = 2$ $d = 4$ magic exceptional supergravity, where the relevant real form is $G = E_{7(-25)}$.

As the first step we need to construct the Lie algebra $\mathfrak{e}_{7(-25)}$. To this aim, we are going to follow the technique outlined in Sec. 7 of [1], which is based on the non-symmetric “mixed” magic square [2, 3, 4]:

Table 1: The “mixed” magic square

	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}
\mathbb{R}	$SO(3)$	$SU(3)$	$USp(6)$	$F_{4(-52)}$
\mathbb{C}	$SU(3)$	$SU(3) \oplus SU(3)$	$SU(6)$	$E_{6(-78)}$
\mathbb{H}_S	$Sp(6, \mathbb{R})$	$SU(3, 3)$	$SO^*(12)$	$E_{7(-25)}$
\mathbb{O}_S	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

The rows and the columns contain the division algebras of the real numbers \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} , the octonions \mathbb{O} and their split forms \mathbb{H}_S and \mathbb{O}_S .

Then the Tits formula gives the Lie algebra \mathcal{L} corresponding to row \mathbb{A} and column \mathbb{B} as [4]:

$$\mathcal{L}(\mathbb{A}, \mathbb{B}) = \text{Der}(\mathbb{A}) \oplus \text{Der}(\mathfrak{J}_3(\mathbb{B})) \dot{+} (\mathbb{A}' \otimes \mathfrak{J}_3'(\mathbb{B})). \quad (1)$$

Here, the symbol \oplus denotes direct sum of algebras, whereas $\dot{+}$ stands for direct sum of vector spaces. Furthermore, Der means the linear derivations, $\mathfrak{J}_3(\mathbb{B})$ denotes the rank-3 Jordan algebra on \mathbb{B} , and the priming amounts to considering only traceless elements. One of the main ingredients entering in the last term is the Lie product, which extends the multiplication to $\mathbb{A}' \otimes \mathfrak{J}_3'(\mathbb{B})$. Its explicit expression for $\mathbb{A} = \mathbb{H}_S$ and $\mathbb{B} = \mathbb{O}$ can be found *e.g.* in [5].

For the Lie algebra of $E_{7(-25)}$ the Tits formula (1) yields:

$$\mathfrak{e}_{7(-25)} = \mathcal{L}(\mathbb{H}_S, \mathbb{O}) = \text{Der}(\mathbb{H}_S) \oplus \text{Der}(\mathfrak{J}_3(\mathbb{O})) \dot{+} (\mathbb{H}_S' \otimes \mathfrak{J}_3'(\mathbb{O})) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{f}_4 \dot{+} (\mathbb{H}_S' \otimes \mathfrak{J}_3'(\mathbb{O})). \quad (2)$$

The second step is to identify the subalgebra \mathfrak{K} generating the maximal compact subgroup $K := (E_{6(-78)} \times U(1))/\mathbb{Z}_3$ of $E_{7(-25)}$. This can be achieved by using the Tits formula (1) once more to compute the manifestly \mathfrak{f}_4 -covariant expression for $\mathfrak{e}_{6(-78)}$:

$$\mathfrak{e}_{6(-78)} = \mathcal{L}(\mathbb{C}, \mathbb{O}) = \mathcal{L}(\mathbb{R}, \mathbb{O}) \dot{+} (i \otimes \mathfrak{J}_3'(\mathbb{O})) = \text{Der}(\mathfrak{J}_3(\mathbb{O})) \dot{+} (i \otimes \mathfrak{J}_3'(\mathbb{O})) = \mathfrak{f}_4 \dot{+} (i \otimes \mathfrak{J}_3'(\mathbb{O})), \quad (3)$$

where we are picking the only imaginary unit $i \in \mathbb{H}_S$ which satisfies $i^2 = -1$. Thus, we obtain:

$$\mathfrak{K} = \text{ad}_i \oplus \text{Der}(\mathfrak{J}_3(\mathbb{O})) \dot{+} (i \otimes \mathfrak{J}_3'(\mathbb{O})), \quad (4)$$

with $\text{ad}_i \in \mathbb{H}_S$ the *adjoint action* of i , generating the maximal compact subgroup $U(1)$ of the group $SL(2, \mathbb{R})$ appearing in (2).

An explicit construction of the matrices ϕ_I , $I = 1, \dots, 78$, realizing the $\mathfrak{e}_{6(-78)}$ subalgebra in its irreducible representation **Fund** = **27** has been performed *e.g.* in Sec. 2.1 of [6] by making use of (3) and of the explicit expression of $\mathfrak{f}_{4(-52)}$ in its irrep. **Fund** = **26** previously computed in [7].

Finally, by putting together all these ingredients, we find that an explicit symplectic realization of the Lie algebra $\mathfrak{e}_{7(-25)}$ in its irreducible representation **Fund** = **56** is as follows [8].

The generators of the maximal compact subgroup K (antihermitian matrices):

$$\mathfrak{e}_{6(-78)} : \quad Y_I = \begin{pmatrix} \phi_I & \vec{0}_{27} & 0_{27} & \vec{0}_{27} \\ \vec{0}_{27}^T & 0 & \vec{0}_{27}^T & 0 \\ 0_{27} & \vec{0}_{27} & -\phi_I^T & \vec{0}_{27} \\ \vec{0}_{27}^T & 0 & \vec{0}_{27}^T & 0 \end{pmatrix}, \quad I = 1, \dots, 78; \quad (5)$$

$$u(1) : Y_{79} = \left(\begin{array}{c|c|c|c} \frac{i}{\sqrt{6}}I_{27} & \vec{0}_{27} & 0_{27} & \vec{0}_{27} \\ \hline \vec{0}_{27}^T & -i\sqrt{\frac{3}{2}} & \vec{0}_{27}^T & 0 \\ \hline 0_{27} & \vec{0}_{27} & -\frac{i}{\sqrt{6}}I_{27} & \vec{0}_{27} \\ \hline \vec{0}_{27}^T & 0 & \vec{0}_{27}^T & i\sqrt{\frac{3}{2}} \end{array} \right); \quad (6)$$

The generators of the coset $\mathcal{M} = G/K$ (hermitian matrices):

$$Y_{\alpha+79} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \vec{0}_{27} & 2iA_\alpha & i\sqrt{2}\vec{e}_\alpha \\ \hline \vec{0}_{27}^T & 0 & i\sqrt{2}\vec{e}_\alpha^T & 0 \\ \hline -2iA_\alpha & -i\sqrt{2}\vec{e}_\alpha & 0_{27} & \vec{0}_{27} \\ \hline -i\sqrt{2}\vec{e}_\alpha^T & 0 & \vec{0}_{27}^T & 0 \end{array} \right), \quad \alpha = 1, \dots, 27; \quad (7)$$

$$Y_{\alpha+106} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \vec{0}_{27} & -2A_\alpha & \sqrt{2}\vec{e}_\alpha \\ \hline \vec{0}_{27}^T & 0 & \sqrt{2}\vec{e}_\alpha^T & 0 \\ \hline -2A_\alpha & \sqrt{2}\vec{e}_\alpha & 0_{27} & \vec{0}_{27} \\ \hline \sqrt{2}\vec{e}_\alpha^T & 0 & \vec{0}_{27}^T & 0 \end{array} \right), \quad \alpha = 1, \dots, 27. \quad (8)$$

Here I_n is the $n \times n$ identity matrix, 0_{27} is the 27×27 null matrix, $\vec{0}_n$ is the zero vector in \mathbb{R}^n , and \vec{e}_α , $\alpha = 1, \dots, 27$, is the canonical basis of \mathbb{R}^{27} .

The matrices A_α are defined in terms of the d -tensor of the **27** of $E_{6(-78)}$. There is a cubic form, which is defined for any $j_1, j_2, j_3 \in \mathfrak{J}_3(\mathbb{O})$ as [9, 10, 11]:

$$Det(j_1, j_2, j_3) := \frac{1}{3} \text{Tr}(j_1 \circ j_2 \circ j_3) + \frac{1}{6} \text{Tr}(j_1) \text{Tr}(j_2) \text{Tr}(j_3) - \frac{1}{6} \left(\text{Tr}(j_1) \text{Tr}(j_2 \circ j_3) + \text{cyclic perm.} \right), \quad (9)$$

where \circ is the product in $\mathfrak{J}_3(\mathbb{O})$. By choosing a basis $\{j_a\}_{a=1, \dots, 26}$ of $\mathfrak{J}_3'(\mathbb{O})$ normalized as $\langle j_a, j_b \rangle := \text{Tr}(j_a \circ j_b) = 2\delta_{ab}$, a completion to a basis for $\mathfrak{J}_3(\mathbb{O})$ can be obtained by adding $j_{27} = \sqrt{\frac{2}{3}}I_3$. Then the matrices A_α 's are 27×27 symmetric matrices, whose components, explicitly computed in [5], satisfy the following relation [9]:

$$(A_\alpha)^\beta_\gamma = \frac{3}{2} Det(j_\alpha, j_\gamma, j_\beta) =: \frac{1}{\sqrt{2}} d_{\alpha\gamma\beta}, \quad (10)$$

where $d_{\alpha\gamma\beta} = d_{(\alpha\gamma\beta)}$ is the totally symmetric rank-3 invariant d -tensor of the **27** of $E_{6(-78)}$, with a normalization suitable to match $Det(j_\alpha, j_\gamma, j_\beta)$ given by (9). Whenever the choice of the basis $\{j_\alpha\}$ is exploited in order to distinguish the identity matrix from the traceless ones, the $d_{\alpha\beta\gamma}$ of E_6 has a maximal manifestly $F_{4(-52)}$ -invariance only. However, it is crucial to point out that, being expressed only in terms of the invariant d -tensor, the result (10) does not depend on the particular choice of the basis $\{j_\alpha\}$. Thus, the expressions of $Y_{\alpha+79}$ (7) and of $Y_{\alpha+106}$ (8) exhibit the maximal manifest compact $[(E_6 \times U(1))/\mathbb{Z}_3]$ -covariance.

A couple of remarks on the properties of the matrices Y_A 's are in order. The first is that they satisfy:

$$Y_A \in \mathfrak{usp}(28, 28), \quad A = 1, \dots, 133. \quad (11)$$

Moreover, in order to guarantee that the period of the maximal torus in the E_6 subgroup equals 4π , the standard choice for the period of the spin representations of the orthogonal subgroups [7, 6], the matrices

Y_A 's are orthonormalized as $\langle Y, Y' \rangle_{56} := \frac{1}{12} \text{Tr}(Y Y')$ with signature $(-^{79}, +^{54})$. As a consequence, the components $(A_\alpha)^\beta_\gamma := A_{\alpha\beta\gamma}$ are normalized as $A_{\alpha\beta\gamma} A^{\eta\beta\gamma} = 5\delta_\alpha^\eta$.

This is consistent with the normalization of the d -tensor (of $E_{6(-26)}$) adopted *e.g.* in [12], which is dictated by the expression $f(z) := \frac{1}{3!} d_{\alpha\beta\gamma} z^\alpha z^\beta z^\gamma$ for the Kähler-invariant $((X^0)^2\text{-rescaled})$ holomorphic prepotential function characterizing special Kähler geometry (see *e.g.* [13, 14, 15], and Refs. therein).

2 Manifestly $[(E_6 \times U(1))/\mathbb{Z}_3]$ -covariant Construction of the Coset \mathcal{M}

In this Section we construct a manifestly $[(E_6 \times U(1))/\mathbb{Z}_3]$ -covariant parametrization of the symmetric space $\mathcal{M} = \frac{E_{7(-25)}}{(E_{6(-78)} \times U(1))/\mathbb{Z}_3}$. As we have seen in the previous Sec. 1, it is generated by the matrices Y_{79+I} , (7) and (8) with $I = 1, \dots, 54$. Through the exponential mapping, it can be defined as follows:

$$\mathcal{M} := \exp \left(\sum_{\alpha=1}^{27} x_\alpha Y_{106+\alpha} + y_\alpha Y_{79+\alpha} \right), \text{ with } x_\alpha \in \mathbb{R}, y_\alpha \in \mathbb{R}, \text{ for } \alpha = 1, \dots, 27. \quad (12)$$

In order to make the complex structure of \mathcal{M} manifest, it is convenient to introduce the following complex linear combinations of the matrices:

$$\zeta_\alpha := \frac{1}{\sqrt{2}} (Y_{79+\alpha} + i Y_{106+\alpha}), \quad \bar{\zeta}_\alpha := \frac{1}{\sqrt{2}} (Y_{79+\alpha} - i Y_{106+\alpha}) \quad (13)$$

together with the corresponding complex linear combinations of the parameters:

$$z_\alpha := \frac{1}{\sqrt{2}} (y_\alpha + i x_\alpha), \quad \bar{z}_\alpha := \frac{1}{\sqrt{2}} (y_\alpha - i x_\alpha), \quad (14)$$

which allows to rewrite (12) as

$$\mathcal{M} := \exp \left(\sum_{\alpha=1}^{27} \bar{z}_\alpha \zeta_\alpha + z_\alpha \bar{\zeta}_\alpha \right). \quad (15)$$

By introducing the 27 dimensional complex vector $z := \sum_{\alpha=1}^{27} z_\alpha \vec{e}_\alpha$, describing the scalar fields, and the 28×28 matrix $\mathcal{A} := \left(\begin{array}{c|c} -\sqrt{2} \sum_{\alpha=1}^{27} \bar{z}_\alpha A_\alpha & z \\ \hline z^T & 0 \end{array} \right)$, the expression for \mathcal{M} (15) enjoys the simple form:

$$\mathcal{M} := \exp \left(\begin{array}{c|c} 0 & \mathcal{A} \\ \hline \mathcal{A}^\dagger & 0 \end{array} \right) = \left(\begin{array}{c|c} \text{Ch}(\sqrt{\mathcal{A}\mathcal{A}^\dagger}) & \mathcal{A} \frac{\text{Sh}(\sqrt{\mathcal{A}^\dagger \mathcal{A}})}{\sqrt{\mathcal{A}^\dagger \mathcal{A}}} \\ \hline \frac{\text{Sh}(\sqrt{\mathcal{A}\mathcal{A}^\dagger})}{\sqrt{\mathcal{A}\mathcal{A}^\dagger}} \mathcal{A}^\dagger & \text{Ch}(\sqrt{\mathcal{A}^\dagger \mathcal{A}}) \end{array} \right). \quad (16)$$

This is a Hermitian matrix, of the same form as the finite coset representative worked out [16] for the *split* (*i.e.* maximally non-compact) counterpart $\mathcal{M}_{\mathcal{N}=8} = \frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2}$, which is the scalar manifold of *maximal* $\mathcal{N} = 8$, $D = 4$ supergravity, associated to $\mathfrak{J}_3(\mathbb{O}_S)$. However, while $\mathcal{M}_{\mathcal{N}=8}$ is real, because of (11) \mathcal{M} is an element of $USp(28, 28)$.

By using the machinery of *special Kähler geometry* (see *e.g.* [13, 14, 15], and Refs. therein), the symplectic sections defining the *symplectic frame* associated to the coset parametrization introduced above can be directly read from (16):

$$\mathcal{M} =: \left(\begin{array}{c|c} u_i^\Lambda(z, \bar{z}) & v_{i\Lambda}(z, \bar{z}) \\ \hline v^{i\Lambda}(z, \bar{z}) & u_\Lambda^i(z, \bar{z}) \end{array} \right), \quad (17)$$

where the symplectic index $\Lambda = 0, 1, \dots, 27$ (with 0 pertaining to the $\mathcal{N} = 2$, $D = 4$ graviphoton), and $i = \bar{\alpha}, 28$. Thus, the symplectic sections read (see *e.g.* [17, 15] and Refs. therein; subscript “28” omitted):

$$f_i^\Lambda : = \frac{1}{\sqrt{2}}(u + v)_i^\Lambda = \left(\bar{f}_\alpha^\Lambda, f^\Lambda \right) := \left(\bar{\mathcal{D}}_\alpha L^\Lambda, L^\Lambda \right) = \exp\left(\frac{1}{2}K\right) \left(\bar{\mathcal{D}}_\alpha X^\Lambda, X^\Lambda \right); \quad (18)$$

$$h_{i\Lambda} : = -\frac{i}{\sqrt{2}}(u - v)_{i\Lambda} = (\bar{h}_{\alpha\Lambda}, h_\Lambda) := (\bar{\mathcal{D}}_\alpha M_\Lambda, M_\Lambda) = \exp\left(\frac{1}{2}K\right) (\bar{\mathcal{D}}_\alpha F_\Lambda, F_\Lambda), \quad (19)$$

where \mathcal{D} is the Kähler-covariant differential operator,

$$\mathcal{V} := (L^\Lambda, M_\Lambda)^T = \exp\left(\frac{1}{2}K\right) (X^\Lambda, F_\Lambda)^T \quad (20)$$

is the symplectic vector of Kähler-covariantly holomorphic sections, and

$$K := -\ln \left[i \left(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda \right) \right] \quad (21)$$

is the Kähler potential determining the corresponding geometry. A more explicit expression for (16) would be needed in order to check that the prepotential F does not exist (i.e., $2F = X^\Lambda F_\Lambda = 0$ [18]) in the symplectic frame we have just introduced, which can be considered the analogue of the Calabi-Vesentini basis [19, 18], whose manifest covariance is the maximal one.

3 The Iwasawa Decomposition and the role of triality

Now we are going to find another parametrization for the coset \mathcal{M} , provided by the Iwasawa decomposition. In this case the maximal manifest covariance is broken down to a subgroup $SO(8)$, thus providing a manifestly triality-symmetric description.

The manifold \mathcal{M} has rank 3, which means that the maximal dimension of the intersection between a Cartan subalgebra of $E_{7(-25)}$ and the generators of \mathcal{M} is 3. In particular, we can pick 3 such generators to be the diagonal generators of the Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ itself, namely $h_1 = Y_{123}$, $h_2 = Y_{132}$ and $h_3 = Y_{133}$.

The following step is to determine a basis \mathcal{W}_+ of $54 - 3 = 51$ positive roots λ_i^+ , $i = 1, \dots, 51$ with respect to \mathfrak{H}_3 . Then the Iwasawa decomposition of the coset \mathcal{M} is defined as:

$$\mathcal{M} := \exp(x_1 h_1 + x_2 h_2 + x_3 h_3) \exp\left(\sum_{i=1}^{51} y_i \lambda_i^+\right). \quad (22)$$

As anticipated, one of its main features is that since the elements $h_1, h_2, h_3 \in \mathfrak{h}_3$ commute with a 28-dimensional subalgebra $\mathfrak{so}(8)$, the Iwasawa parametrization of \mathcal{M} exhibits a maximal manifest covariance given by $SO(8)$. Therefore, the 51-dimensional linear space Λ_+ generated by the positive roots \mathcal{W}_+ is invariant under the (adjoint) action of $SO(8)$, and it decomposes into irreps. of $SO(8)$ as:

$$\Lambda_+ = \mathbf{1}^3 + \mathbf{8}_v^2 + \mathbf{8}_c^2 + \mathbf{8}_s^2, \quad (23)$$

which is a manifestly triality-symmetric decomposition. In particular, at the level of algebras $\mathfrak{so}(8) = \mathfrak{tri}(\mathbb{O})$ with the automorphism group $\text{Aut}(\mathfrak{t}(\mathbb{O})) = \text{Spin}(8)$ of the normed triality over the octonions \mathbb{O} [20].

It is worth remarking that the appearance of the square for the three **8** irreps. in (23) is a consequence of the complex (special Kähler) structure of the coset \mathcal{M} .

Moreover, it should be observed that the $SO(8)$ entering in (23) can be identified as:

$$SO(8) \subset [(SO(10) \times U(1)) \cap F_{4(-52)}]. \quad (24)$$

This can be understood by noticing that it can be obtained from both the following chains of maximal symmetric embeddings [21]:

$$E_{7(-25)} \supset E_{6(-78)} \times U(1)' \supset SO(10) \times U(1)' \times U(1)'' \supset SO(8) \times U(1)' \times U(1)'' \times U(1)''' \quad (25)$$

and

$$E_{7(-25)} \supset E_{6(-78)} \times U(1)' \supset F_{4(-52)} \times U(1)' \supset SO(9) \times U(1)' \supset SO(8) \times U(1)'. \quad (26)$$

In the last line of (25) the first two $U(1)$ factors have the physical meaning of “extra” T -dualities generated by the Kaluza-Klein reductions, respectively $D = 5 \rightarrow D = 4$, and $D = 6 \rightarrow D = 5$.

Denoting with subscripts $U(1)$ -charges, the adjoint irrep. **133** of $E_{7(-25)}$ branches according to (25) as (see e.g. [22]):

$$\begin{aligned} \mathbf{133} &= \mathbf{78}_0 + \mathbf{1}_0 + \mathbf{27}_{-2} + \mathbf{27}'_{+2} \\ &= \mathbf{1}_{0,0} + \mathbf{16}_{0,-3} + \mathbf{16}'_{0,+3} + \mathbf{45}_{0,0} + \mathbf{1}_{0,0} \\ &\quad + \mathbf{1}_{-2,+4} + \mathbf{10}_{-2,-2} + \mathbf{16}_{-2,+1} \\ &\quad + \mathbf{1}_{+2,-4} + \mathbf{10}_{+2,+2} + \mathbf{16}'_{+2,-1} \\ &= \mathbf{1}_{0,0,0} + \mathbf{8}_{c,0,-3,1} + \mathbf{8}_{s,0,-3,-1} + \mathbf{8}_{c,0,+3,-1} + \mathbf{8}_{s,0,+3,+1} \\ &\quad + \mathbf{1}_{0,0,0} + \mathbf{8}_{v,0,0,+2} + \mathbf{8}_{v,0,0,-2} + \mathbf{28}_{0,0,0} + \mathbf{1}_{0,0,0} \\ &\quad + \mathbf{1}_{-2,+4,0} + \mathbf{1}_{-2,-2,+2} + \mathbf{1}_{-2,-2,-2} + \mathbf{8}_{v,-2,-2,0} + \mathbf{8}_{c,-2,+1,+1} + \mathbf{8}_{s,-2,+1,-1} \\ &\quad + \mathbf{1}_{+2,-4,0} + \mathbf{1}_{+2,+2,-2} + \mathbf{1}_{+2,+2,+2} + \mathbf{8}_{v,+2,+2,0} + \mathbf{8}_{c,+2,-1,-1} + \mathbf{8}_{s,+2,-1,+1}. \end{aligned} \quad (27)$$

4 Final Remarks

It is very interesting to remark that being based only on the algebraic properties of the “mixed” Freudenthal-Tits magic square in Table 1, the construction of the basis with the maximal possible covariance (16) and the computation of the Iwasawa decomposition (22) described here can be both generalized [8] at least to a broader class of minimally non-compact, simple groups of type E_7 [23]. Moreover, it also turns out that, like for \mathcal{M} , in all these cases the maximal covariance (at least at the Lie algebra level) of the Iwasawa decomposition is given by the automorphism algebra of the corresponding normed triality [20].

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